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THE DENSEST HEMISPHERE PROBLEM

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Abstract. Given a set K of n points on the unit sphere S^d in d-dimensional Euclidean space, a hemisphere of S^d is densest if it contains a largest subset of K. In this paper we consider the problem of determining a densest hemisphere and present the following complementary results: (i) a discretized version of the original problem, restated as a feasibility question, is NP-complete when both n and d are arbitrary; (ii) when the number d of dimensions is fixed, there exists a polynomial time algorithm which solves the problem in time $O(n^{d-1} \log n)$ on a random access machine with unit cost arithmetic operations.

1. Introduction

This paper is motivated by the following simple geometric problem: let $\mathbf{R}^d$ be the d-dimensional Euclidean space and let S^d be the sphere of unit radius with center at the origin of $\mathbf{R}^d$. Let K be a set of n points on S^d. Find a hemisphere of S^d which contains a largest subset of K.

This geometric problem was posed to the authors by H. S. Witsenhausen for its relevance to applications of statistical analysis and operations research. It was apparently originated by J. B. Kadane and R. Friedheim as a formalization of the following situation in political science. The coordinates of the points in K correspond to preferences of n voters on d relevant political issues; the axis of the maximizing hemisphere then corresponds to a position on these issues which is likely to be supported by a majority of the voters.[2]

In thinking about such applications, it is more convenient to formulate the problem in terms of vectors and inner products. (This will also enable us to make

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[2] Private communication of J.B Kadane, Department of Statistics, Carnegie-Mellon University.

the useful restriction that all coordinates are national numbers, thus placing the problem in the standard discrete form to which computational complexity arguments can be applied.)

To be specific, let $K = \{P_1, P_2, \ldots, P_n\}$ be a finite subset of $\mathbf{Q}^d$, where, as usual, $\mathbf{Q}$ is the set of rationals. There are actually two parallel problems to consider:

CLOSED HEMISPHERE: Find that $x \in \mathbf{R}^d$ such that $|x| > 0$ and $|\{P \in K: x \cdot P \geq 0\}|$ is maximized.

OPEN HEMISPHERE: Find that $x \in \mathbf{R}^d$ such that $|\{P \in K: x \cdot P > 0\}|$ is maximized.

The correspondence with the geometric problem comes from the fact that each $x \in \mathbf{R}^d$ determines a hyperplane through the origin $\{y \in \mathbf{R}^d: y \cdot x = 0\}$ which partitions S^d into the two open hemispheres $\{y \in S^d: y \cdot x < 0\}$ and $\{y \in S^d: y \cdot x > 0\}$. However, observe that the vector problem is in a sense more general as it allows more than one point along a single ray from the origin.

In this paper we present the following results: Both the CLOSED and OPEN HEMISPHERE problems are NP-complete if the number of dimensions is not fixed in advance (Section 2). This means that there can be no polynomial time algorithm for the general problems unless many other famous intractable problems also have polynomial time algorithms, an unlikely event [2, 3]. Interestingly, however, as we shall see in Sections 3 and 4, a densest hemisphere can be algorithmically determined for fixed d in time $O(n^{d-1} \log n)$, where the adopted computation model is the random access machine of [2], with all arithmetic operations having unit cost.[1] The latter result not only shows that the problem can be solved in polynomial time for fixed d, but it also provides an attractive method for cases in which d is a small integer, say 4 or less.

It may be pointed out that the presented algorithm can be modified to solve interesting variants of the problem, such as the determination of a densest hemisphere when each point in K has an assigned weight. Another variant of the problem, discussed by Reiss and Dobkin [1], is to determine if there is a hemisphere which contains the entire set K. This variant, however, has been shown to be equivalent to linear programming and may well be simpler than the general problem discussed in this paper.

2. NP-completeness of the HEMISPHERE problems

In this section we present a proof that CLOSED HEMISPHERE, stated as a feasibility question, is NP-complete. (The construction in addition shows that the

[1] For models in which the unit of time is a bit operation and hence arithmetic operations have costs depending upon the lengths of the operands, the above running time bound would be multiplied by a factor depending on these lengths, but would still be a polynomial.

OPEN problem is NP-complete.) The statement of the problem as a feasibility question goes as follows.

HEMISPHERE. Given positive integers d and M and a finite set $K \subseteq \mathbf{Q}^d$, does there exist a $P^* \in \mathbf{R}^d$ such that $|P^*| > 0$ and $|\{P \in K : P \cdot P^* \geq 0\}| \geq M$?

To prove that this problem is NP-complete, we must (i) show that it can be solved non-deterministically in polynomial time, and (ii) reduce a known NP-complete problem to it [2, 3]. For the former, we observe that if such a P^* exists, another one could be found as the solution to a linear programming problem involving the set $\{P \in K : P \cdot P^* \geq 0\}$, and hence must have rational coordinates of polynomially bounded length. Thus, all we have to do is guess these coordinates.

To complete the NP-completeness proof for HEMISPHERE, we reduce the NP-complete MAXIMUM 2-SATISFIABILITY problem [4] to it.

MAXIMUM 2-SATISFIABILITY (MAX 2-SAT)

Given: positive integers m and $N > 1$

finite collection $\mathscr{C}$ of two-element subsets of

$X = \{x_1, \bar{x}_1, x_2, \bar{x}_2, \ldots, x_m, \bar{x}_m\}$ such that $|\mathscr{C}| \geq N$.

Question: does there exist a subset $X' \subseteq X$ with $|X' \cap \{x_i, \bar{x}_i\}| = 1$

for $1 \leq i \leq m$ such that $|\{c \in \mathscr{C} : X' \cap c \neq \emptyset\}| \geq N$?

We shall show how to transform any instance of MAX 2-SAT to a corresponding instance of HEMISPHERE in polynomial time, in such a way that the answer for the second instance is affirmative if and only if the answer for the first instance is also affirmative.

In what follows, we shall use a shorthand notation for sets of vectors. If $a, n \in \mathbf{Z}$ and $n \geq 0$, we let $(a)^n$ stand for the set consisting of the single n-dimensional vector ("n-tuple") $\langle a, a, \ldots, a\rangle$, all of whose components are a. If $S \subseteq \mathbf{Z}$ is a finite set, S^n will represent the set of all possible n-tuples with components from the set S. (Observe that $|S^n| = |S|^n$.) Finally, if U is a set of n-tuples and V is a set of m-tuples, UV is a set of $|U| \cdot |V|$ $(n+m)$-tuples of the form $\langle a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m\rangle$ where $\langle a_1, a_2, \ldots, a_n\rangle \in U$ and $\langle b_1, b_2, \ldots, b_m\rangle \in V$.

We now describe our construction. Suppose m, N, and $\mathscr{C}$ provide an instance of MAX 2-SAT. Let $|\mathscr{C}| = s$ and $t = \lceil \log_2(ms+1)\rceil$. We construct three sets A, B, and C of d-dimensional vectors, with $d = m + 1 + 3t$, specified as follows.

The set A will consist of $2m \cdot 2^{3t} \leq 2m(2ms+1)^3$ vectors, subdivided into subsets A_i and $\bar{A}_i$, $1 \leq i \leq m$, where

$$A_i = (0)^{i-1}(1)(0)^{m-i}(1)\{1, -1\}^{3t},$$

and

$$\bar{A}_i = (0)^{i-1}(-1)(0)^{m-i}(1)\{1, -1\}^{3t}.$$

The set B will consist of $2m \cdot 2^t \leq 2m(2ms+1)$ vectors, subdivided into subsets B_i and $\bar{B}_i$, $1 \leq i \leq m$, where

$$B_i = (0)^{i-1}(4)(0)^{m-i}(-2)(0)^{2t}\{1, -1\}^t,$$

and

$$\bar{B}_i = (0)^{i-1}(-4)(0)^{m-i}(-2)(0)^{2t}\{1,-1\}^t.$$

Finally, the set C consists of one representative for each $c \in \mathscr{C}$, constructed as follows. Denoting $x_j[1] = x_j$ and $x_j[-1] = \bar{x}_j$, the two-element subset $c = \{x_i[e_i], x_j[e_j]\} \in \mathscr{C}$, with $1 \leq i < j \leq m$ and $e_i, e_j \in \{1, -1\}$ corresponds to the vector in C

$$P = (0)^{i-1}(4e_i)(0)^{j-i}(4e_j)(0)^{m-j}(1)(0)^{3t}.$$

The instance of HEMISPHERE corresponding to m, N, and $\mathscr{C}$ is then given by

$$d = m+1+3t, \quad K = A \cup B \cup C, \quad \text{and} \quad M = 2m \cdot 2^{3t} + m \cdot 2^t + N.$$

Clearly, given m, N, and $\mathscr{C}$, this instance can be constructed in time polynomial in the parameters m and s (and, clearly, it is an instance of HEMISPHERE). Thus, all that remains is to show that the desired X' for the MAX 2-SAT problem exists if and only if the desired P^* for the HEMISPHERE problem also exists.

Assume first that the desired X' exists; that is, there is an $X' \subseteq \{x_1, \bar{x}_1, x_2, \bar{x}_2, \ldots, x_m, \bar{x}_m\}$ such that $|X' \cap \{x_i, \bar{x}_i\}| = 1$, $1 \leq i \leq m$, and $|\{c \in \mathscr{C}: X' \cap c \neq \emptyset\}| \geq N$. The desired $P^* = \langle p_1^*, p_2^*, \ldots, p_d^* \rangle$ is then given by

$$p_j^* = \begin{cases} 1 & \text{if } 1 \leq j \leq m \text{ and } x_j \in X', \\ -1 & \text{if } 1 \leq j \leq m \text{ and } \bar{x}_j \in X', \\ 1.5 & \text{if } j = m+1, \\ 0 & \text{if } m+2 \leq j \leq d. \end{cases}$$

The reader may readily verify that

$$|\{P \in A: P^* \cdot P \geq 0\}| = |A| = 2m \cdot 2^{3t},$$

$$|\{P \in B: P^* \cdot P \geq 0\}| = |B|/2 = m \cdot 2^t,$$

and

$$|\{P \in C: P^* \cdot P \geq 0\}| \geq N.$$

Hence, $|\{P \in K = A \cup B \cup C: P^* \cdot P \geq 0\}| \geq 2m \cdot 2^{3t} + m \cdot 2^t + N = M$, and so P^* has the desired properties.

Now suppose $P^* = \langle p_1^*, p_2^*, \ldots, p_d^* \rangle$ is a vector having the desired properties for d, K, and M. Then it must also obey the following claims, which will lead us to the desired X'. For convenience, let $A^+ \stackrel{\Delta}{=} \{P \in A: P^* \cdot P \geq 0\}$ and let B^+ and C^+ be analogously defined.

Claim 1. $|A^+| > 2m \cdot 2^{3t} - 2^{2t}$.

By assumption, $M \leq |A^+| + |B^+| + |C^+|$, whence $|A^+| \geq M - |B^+| - |C^+|$. But $|B^+| \leq |B| = 2m \cdot 2^t$ and $|C^+| \leq |C| = s$, whence $|A^+| \geq (2m \cdot 2^{3t} + m \cdot 2^t + N) - 2m \cdot 2^t - s > 2m \cdot 2^{3t} - m \cdot 2^t - s$. Since $s < ms + 1 \leq 2^t$ and $m \leq ms \leq 2^t - 1$, the claim follows. $\square$

Claim 2. $p^*_{m+1} > 0$.

First suppose $p^*_{m+1} < 0$, and consider the bijection (i.e., the pairing) $f: A \to A$ defined by $f(\langle p_1, p_2, \ldots, p_d \rangle) = \langle q_1, q_2, \ldots, q_d \rangle$, where

$$q_j = \begin{cases} -p_j & 1 \leqslant j \leqslant m \text{ or } m+2 \leqslant j \leqslant d, \\ 1 & j = m+1. \end{cases}$$

From the definition of f we have $P + f(P) = (0)^m(2)(0)^{3t}$ for all $P \in A$. Thus, by our assumption that $p^*_{m+1} < 0$, we have $P^* \cdot (P + f(P)) < 0$ for all $P \in A$, and hence $P^* \cdot P \geqslant 0$ implies $P^* \cdot f(P) < 0$. This means

$$|\{P \in A: P^* \cdot P \geqslant 0\}| \leqslant |\{P \in A: P^* \cdot P < 0\}|,$$

a contradiction to Claim 1. Thus we must have $p^*_{m+1} \geqslant 0$. Suppose $p^*_{m+1} = 0$. By the requirements of the HEMISPHERE problem, P^* must have at least one non-zero component, say p^*_k. Let

$$A' = \{P \in A: p_k \cdot p^*_k < 0\}.$$

By the definition of A, we must have $|A'| \geqslant 2^{3t} > 2 \cdot 2^{2t}$. Consider the bijection $g: A' \to A'$ defined by $g(\langle p_1, p_2, \ldots, p_d \rangle) = \langle q_1, q_2, \ldots, q_d \rangle$, where

$$q_j = \begin{cases} -p_j & 1 \leqslant j \leqslant d \text{ and } j \neq \{k, m+1\}, \\ p_j & j = k, \\ 1 & j = m+1. \end{cases}$$

From the definition we have $P^* \cdot (P + g(P)) = 2p_k \cdot p^*_k + 2p^*_{m+1} = 2p_k \cdot p^*_k < 0$, since we are assuming $p^*_{m+1} = 0$. Thus, at least half of the vectors in A' have negative dot products with P^*, and hence

$$|\{P \in A: P^* \cdot P \geqslant 0\}| \leqslant |A| - |A'|/2 < 2m \cdot 2^{3t} - 2^{2t}$$

in violation of Claim 1. Thus we must have $p^*_{m+1} > 0$, as claimed. □

Claim 3. *For all i, $1 \leqslant i \leqslant m$, $|\{P \in B_i \cup \bar{B}_i: P^* \cdot P \geqslant 0\}| \leqslant 2^t$.*

For each i, $1 \leqslant i \leqslant m$, consider the bijection $h: B_i \cup \bar{B}_i \to B_i \cup \bar{B}_i$ defined by $h(\langle p_1, p_2, \ldots, p_d \rangle) = \langle q_1 . q_2, \ldots, q_d \rangle$, where

$$q_j = \begin{cases} -p_j & 1 \leqslant j \leqslant m \text{ or } m+2 \leqslant j \leqslant d, \\ p_j & j = m+1. \end{cases}$$

From the definition, we must have $P + h(P) = (0)^m(-4)(0)^{3t}$ for all $P \in B_i \cup \bar{B}_i$. Thus Claim 2 implies that $P^* \cdot (P + H(P)) < 0$ for all $P \in B_i \cup \bar{B}_i$, and so $|\{P \in B_i \cup \bar{B}_i: P^* \cdot P \geqslant 0\}| \leqslant |B_i \cup \bar{B}_i|/2 = 2^t$, proving the claim. □

For convenience let T denote the set of integers $\{m+2+2t, m+3+2t, \ldots, m+1+3t\}$.

Claim 4. $p^*_{m+1} \geq \sum_{i \in T} |p^*_i|$.

Define the set

$$A'' = \{P \in A\colon \text{for all } i \in T, p_i \cdot p^*_i \leq 0\}$$

and notice that $|A''| = 2m \cdot 2^{2t}$. Consider the bijection $k: A'' \to A''$ defined by $k(\langle p_1, p_2, \ldots, p_d \rangle) = \langle q_1, q_2, \ldots, q_d \rangle$, where

$$q_j = \begin{cases} p_j & j = m+1 \text{ or } j \in T, \\ -p_j & 1 \leq j \leq m \text{ or } m+2 \leq j \leq m+1+2t. \end{cases}$$

It is not difficult to see that for all $P \in A''$, $P^* \cdot (P + k(P)) = 2p^*_{m+1} - 2\sum_{i \in T} |p^*_i|$. If $p^*_{m+1} < \sum_{i \in T} |p^*_i|$, then we would have

$$|\{P \in A''\colon P^* \cdot P < 0\}| > \frac{|A''|}{2} = 2m \cdot 2^{2t}/2,$$

whence, $|A^+| < |A| - |A''/2| = 2m \cdot 2^{3t} - 2m \cdot 2^{2t-1}$. This and Claim 3 would imply

$$\begin{aligned} |A^+| + |B^+| + |C^+| &< (2m \cdot 2^{3t} - m \cdot 2^{2t}) + m \cdot 2^t + s \\ &= M - m \cdot 2^{2t} + s - N \\ &\leq M - m \cdot 2^{2t} + 2^t < M, \end{aligned}$$

a contradiction. □

Claim 5. *For each* i, $1 \leq i \leq m$, $|p^*_i| \geq p^*_{m+1}/4$.

By Claim 2 and the definitions of $B_i \cup \bar{B}_i$ and of index set T, we have for all $P \in B_i \cup \bar{B}_i$ that

$$P^* \cdot P \leq 4|p^*_i| - 2p^*_{m+1} + \sum_{j \in T} |p^*_j|.$$

By Claim 4, we have $p^*_{m+1} \geq \sum_{j \in T} |p^*_j|$, whence $P^* \cdot P \leq 4|p^*_i| - p^*_{m+1}$.

If $4|p^*_i| < p^*_{m+1}$, we would have $P^* \cdot P < 0$ for all $P \in B_i \cup \bar{B}_i$, whence

$$|\{P \in B_i \cup \bar{B}_i\colon P^* \cdot P \geq 0\}| = 0$$

and so by Claim 3

$$\begin{aligned} |A^+| + |B^+| + |C^+| &\leq |A^+| + (m-1)2^t + |C^+| \\ &\leq 2m \cdot 2^{3t} + m \cdot 2^t - 2^t + s < M - 2^t + s < M, \end{aligned}$$

yet another contradiction. □

Claim 6. $|C^+| \geq N$.

We must have $|C^+| \geq M - |A^+| - |B^+|$. Using the inequalities $|A^+| \leq |A|$ and $|B^+| \leq m \cdot 2^t$ (by Claim 3) we have $|C^+| \geq (2m \cdot 2^{3t} + m \cdot 2^t + N) - 2m \cdot 2^{3t} - m2^t = N$. □

Claim 7. *The set* $X' \stackrel{\Delta}{=} \{x_i[1]: 1 \leq i \leq m \text{ and } p_i^* > 0\} \cup \{x_i[-1]: 1 \leq i \leq m \text{ and } p_i^* < 0\}$ *is the desired subset of* X.

The set X' is well defined and obeys $|X' \cap \{x_i, \bar{x}_i\}| = 1$, $1 \leq i \leq m$, since by Claim 5, $p_i^* \neq 0$, $1 \leq i \leq m$. Furthermore, we claim that if $P \in C$ is such that $P^* \cdot P \geq 0$, then the two element set $c \in \mathscr{C}$ corresponding to P has nonvoid intersection with X'. For suppose that $c = \{x_i[s_i], x_j[s_j]\}$ for $s_i, s_j \in [1, -1]$ and $1 \leq i < j \leq m$ and let e_i be such that $x_i[e_i] \in X'$, $1 \leq i \leq n$. We then have that $X' \cap c = \emptyset$ if and only if $s_i e_i = s_j e_j = -1$. Now, recalling that $c = \{x_i[s_i], x_j[s_j]\} \leftrightarrow P = (0)^{i-1}(4s_i)(0)^{j-i}(4s_j)(0)^{m-j}(1)(0)^{3t}$ we have that $X' \cap c = \emptyset$ implies, by Claim 5,

$$\begin{aligned} P^* \cdot P &= 4s_i p_i^* + 4s_j p_j^* + p_{m+1}^* = 4s_i e_i |p_i^*| + 4s_j e_j |p_j^*| + p_{m+1}^* \\ &= -4(|p_i^*| + |p_j^*|) + p_{m+1}^* \leq -2p_{m+1}^* + p_{m+1}^* < 0, \end{aligned}$$

whence we conclude that $P^* \cdot P \geq 0$ implies $X' \cap C \neq \emptyset$. Thus, by Claim 6, the set X' satisfies all the conditions of the solution to the MAX 2-SAT problem for a given X, $\mathscr{C}$, and N. □

From the above arguments we conclude that the desired X' exists if and only if the desired P^* exists. Thus we have successfully reduced MAXIMUM 2-SATISFIABILITY to HEMISPHERE, and completed the proof that the latter is NP-complete.

Our proof also shows that the corresponding problem in which we require that $P^* \cdot P$ strictly exceed 0 is NP-complete, as the reader may readily verify. In addition, we note that the set K we constructed had the following property: for all $P \in K$, $|P| > 0$ and $\{\alpha|P|: \alpha \in \mathbf{R} \text{ and } \alpha > 0\} \cap K = \{P\}$. Thus each point of K corresponded to a unique ray from the origin of $\mathbf{R}^d$ and hence to a unique point on S^d. Therefore the geometric versions of our problems are also at least as hard as an NP-complete problem. One final note on our construction is the observation that the set K is contained in $\{-4, -2, -1, 0, 1, 4\}^d$ and hence the complexity of HEMISPHERE does not depend on having arbitrarily complicated coordinates for the members of K.

3. Algorithms for finding densest hemispheres

In this section we shall present algorithms for the CLOSED and OPEN HEMISPHERE problems which run in time $O(n^d)$ when the dimension d is fixed. In the next section we present an $O(n \log n)$ algorithm for the $d = 2$ case, which enables us to speed up the algorithm for $d \geq 2$ to $O(n^{d-1} \log n)$.

The simpler of our two algorithms is the one for the CLOSED HEMISPHERE problem, and we shall consider it first. To provide some concrete intuition, suppose that $d = 3$ and let $K = \{P_1, P_2, \ldots, P_n\}$ be the given set of vectors applied to the origin; also let $H(P_i)$ be the plane through the origin orthogonal to vector P_i. The set of planes $\{H(P_i)$: $1 \leq i \leq n\}$ partitions the space into unbounded cones with vertex in the origin. Clearly, if two points x_1 and x_2 belong to the same cone, then $x_1 \cdot P_i \geq 0 \Leftrightarrow x_2 \cdot P_i \geq 0$, that is, each cone is an equivalence class of points x characterized by a constant value of the function $A'(x) = |\{P \in K: x \cdot P \geq 0\}|$. Suppose that $A'(x)$ is maximized in some cone $\mathcal{R}$. Then it must be maximized on the face of this cone. To find this maximum value it is sufficient that we explore the faces of $\mathcal{R}$, which are contained in planes of the set $\{H(P_i)$: $1 \leq i \leq n\}$. Thus there are planes in this set such that if we project the set K on any of them and solve the ensuing 2-dimensional CLOSED HEMISPHERE problem, we obtain the solution to our original problem. Unfortunately, this subset is not known *a priori*, so that all the members of $\{H(P_i)$: $1 \leq i \leq n\}$ must be tried. This informally shows that the given 3-dimensional problem can be reduced to a collection of (at most) n 2-dimensional problems of the same type.

We shall now give a more technical description of the algorithm which—as the preceding informal discussion illustrates—is defined recursively. It is also convenient to distinguish the points in K from their coordinates and restate the problem in a slightly generalized form.

Closed hemisphere (CH)

Given integers d and D, with $1 \leq D \leq d$, a finite set $V \subset \mathbf{Q}^d$ such that $T = \{y \in \boldsymbol{R}^d: y \cdot v = 0$ for all $v \in V\}$ is a D-dimensional subspace of $\boldsymbol{R}^d$, and a set $K = \{P_1, P_2, \ldots, P_n\}$ with a map $c: K \to T \cap \mathbf{Q}^d$. Find an $x \in T$, with $|x| > 0$, which maximizes

$$A(x) = |\{P \in K: x \cdot c(P) \geq 0\}|.$$

We say that $[d, D; V; K, c]$ is the *parameter set* of the CH problem.

The closed hemisphere problem, as stated in Section 1, corresponds to CH with $d = D$, $V = \emptyset$, $c(P) = P$ for all $P \in K$.

The CH problem is easily solved in two special cases:

(I) Suppose $c: K \to T \cap \mathbf{Q}^d$ is such that $c(P) = \mathbf{0}$ (the origin of $\boldsymbol{R}^d$) for all $P \in K$. Then choosing any $x \in T$ will maximize $A(x)$. The number of steps required to find such an x depends only on d, so the overall effort required in this case will be $O(nd)$ even if we have to verify that $c(P) = \mathbf{0}$ for all $P \in K$.

(II) Suppose $D = 1$ and case (I) does not hold. Then T is a straight line, and we can find a rational basis vector v such that $T = \{\alpha v: \alpha \in \boldsymbol{R}\}$ in time depending only on d. Given v, we can restrict our attention to just two candidates for x, v and $-v$, and choose the one with largest value of $A(x)$. Again the amount of work will be $O(nd)$, most of the time here spent evaluating $A(x)$.

Now suppose that neither (I) nor (II) applies. We shall show how to reduce the CH problem under consideration to a collection of n or fewer CH problems in

$D-1$ dimensions. Let $U=\{c(P): P\in K\}-\{\mathbf{0}\}$ and for each $u\in U$, let $H(u)=\{y\in T: y\cdot u=0\}$. The hyperplanes $H(u)$ partition T into convex regions. On the interior of each region $A(x)$ is constant, although it may experience a discontinuous increase at region boundaries. Let A^* be the largest value of $A(x)$ for $x\in T$, and, for an extremizing x, let $U'=\{c(P): p\in K, c(P)\neq\mathbf{0}$ and $c(P)\geqslant 0\}$. Clearly, for some $u\in U'$ there exists a $y\in H(u)$ with $|y|>0$ such that $A^*=A(y)$; thus, for such a u, the D-dimensional CH problem can be replaced by a $(D-1)$-dimensional CH problem $[d', D'; V'; K', c']$, whose parameters are so defined

$$d'=d,\qquad D'=D-1,\qquad V'=V\cup\{u\},\qquad K'=K,$$
$$c'(P)=c(P)-\frac{c(P)\cdot u}{|u|^2}u$$

for all $P\in K$. Observe that $T'=\{y\in \boldsymbol{R}^d: y\cdot v=0$ for all $v\in V'\}=H(u)$, $c'(P)$ is merely the projection of $c(P)$ on $H(u)$, and, for all $x\in H(u)$, we have $x\cdot c'(P)=x\cdot(x(P)-(c(P)\cdot u/|u|^2)u)=x\cdot c(P)$. Since the proper choice of u is not known *a priori*, we must try the described reduction for each $c(P)\in U$. This reduces the given CH problem to a collection of at most n CH problems in one less dimensions. We thus obtain a recursive procedure for solving the CH problem in $D=d$ dimensions. The overall running time is at most $O(dn^d)$, as can be seen by standard recurrence relation arguments.

With this background, we are now prepared to consider the more complex OPEN HEMISPHERE problem. Here again we shall present a recursive algorithm, in which a given D-dimensional problem is reduced to a collection of several $(D-1)$-dimensional problems. In contrast to the CH case, however, the reduced problems of an open hemisphere problem are not necessarily of the same type as their parent problem. Therefore it is convenient to define the following composite MIXED HEMISPHERE problem.

Mixed hemisphere (MH)

Given integers d and D with $1\leqslant D\leqslant d$, a finite set $V\subset\mathbf{Q}^d$ such that $T=\{y=\boldsymbol{R}^d: y\cdot v=0$ for all $v\in V\}$ is a D-dimensional subspace of $\boldsymbol{R}^d$, and a set $K=\{P_1, P_2, \ldots, P_n\}$ with maps $c: K\to T\cap\mathbf{Q}^d$ and $s: K\to\{0,1\}$. Find an $x\in T$ which maximizes

$$A(x)=|\{P\in K: s(P)=0 \text{ and } x\cdot c(P)\geqslant 0\}$$
$$\cup\{P\in K: s(P)=1 \text{ and } x\cdot c(P)>0\}|.$$

We say that $[d, D; V; K, c, s]$ is the parameter set of the MH problem.

The open hemisphere problem, as stated in Section 2, corresponds to MH with $d=D$, $V=\emptyset$, $c(P)=P$, and $s(P)=1$ for all $P\in K$.

The crucial difference between the CLOSED and the MIXED HEMISPHERE problems lies in the function $s : K \to \{0, 1\}$ which dichotomizes the set K, and in the fact that the $\mathbf{0}$ vector is in the range of allowable solutions.

In parallel with the previous discussion of the CH problem, the MH problem is easily solved in two special cases, both requiring computational work at most $O(nd)$:

(i) $c : K \to T \cap \mathbf{Q}^d$ is such that $c(P) = \mathbf{0}$ for all $P \in K$. Then any $x \in T$ maximizes $A(x)$ (in particular $x = \mathbf{0}$).

(ii) $D = 1$ and (i) does not hold. Then $T = \{\alpha v : \alpha \in \boldsymbol{R}\}$, and we can restrict ourselves to the three candidates $-v$, v, and $\mathbf{0}$, choosing the one with largest value of $A(x)$.

We now discuss the reduction when neither (i) nor (ii) apply. Let U and $H(u)$, for each $u \in U$, be as previously defined, and let $A^* = \max\{A(x) : x \in T\}$.

Lemma 3.1. *There exists a $u \in U$ and a $y \in H(u)$ such that either*

(1) $A^* = A(y)$, *or*

(2) $A^* = \lim_{\alpha \downarrow 0} A(y + \alpha u)$ *and* $s(P) = 1$ *for some* $P \in K$.

Proof. Suppose (1) does not hold. Then A^* must be realized by some x on the interior R of some closed region $\bar{R}$. Since $x \notin \bar{R} - R$, we have $|x| > 0$. Suppose $s(P) = 0$ for all $P \in K$. Then $A(x) = |\{P \in K : x \cdot c(P) \geq 0\}|$. However, note that $x \in R$ and $u \cdot x \geq 0$ imply $u \cdot z \geq 0$ for all $z \in \bar{R}$, by the definition of R and the continuity of the inner product. This means that for all points $y \in \bar{R}$, $A(y) \geq A(x) = A^*$, a contradiction of our assumption that (1) does not hold. Thus there must exist some $P \in K$ with $s(P) = 1$, as claimed.

We must now show that, if the extremizing x is in the interior R of some region $\bar{R}$ in the partition of T produced by the hyperplanes $H(u)$, then x is of the form $y + \alpha u$, for some $u \in U$, $y \in H(u)$, and $\alpha > 0$. First of all, for any $z \in R$, $A(z) = A^*$. Let F be a face of R; obviously $F \subseteq H(u)$ for some $u \in U$. There is a point $x \in R$ which can be expressed as $(y + \alpha u)$, where y is a point of F (hence $y \in H(u)$) and α is a convenient' chosen real number. All that remains to be shown is that there is at least one such $u \in U$ which yields $\alpha > 0$. Let $U_F = \{u \in U : F \subseteq H(u)\}$. For any $x \in R$ and $u \in U_F$, $u \cdot x \neq 0$. Suppose that for all $u \in U_F$, $x \cdot u < 0$. Let R' be a region of T that shares F as a boundary with R, and let x' be a point on the interior of R'. For all $P \in K$ such that $c(P) \notin U_F$, $c(P) \cdot x < 0$ if and only if $c(P) \cdot x' > 0$ and similarly $c(P) \cdot x = 0$ if and only if $c(P) \cdot x' = 0$. However, by supposition, for all P with $c(P) \in U_F$, $c(P) \cdot x' > 0$ and $c(P) \cdot x < 0$. Since $U_F \neq \emptyset$, this means that $A(x') > A(x)$, a contradiction. Thus, there exists a $u \in U_F$ such that $u \cdot x > 0$, that is, $x \cdot u = (y + \alpha u) \cdot u = y \cdot u + \alpha |u|^2 = \alpha |u|^2 > 0$, and hence $\alpha > 0$. $\square$

This lemma suggests a method for reducing a given D-dimensional MH problem. Since we do not know the vector u, nor whether (1) or (2) holds, for each $u \in U$ we generate two $(D-1)$-dimensional MH subproblems, corresponding to (1) and (2) respectively. In this manner, a given D-dimensional MH problem is replaced by at most $2n$ $(D-1)$-dimensional MH problems, each of which produces a candidate for the solution of the original problem.

Specifically, in the hypothesis that (1) holds for u, the search for y corresponds to the following MH problem with parameters $[d', D'; V'; K', c', s']$:

$$d' = d, \quad D' = D = 1, \qquad V' = V \cup \{u\}, \qquad K' = K,$$

$$c'(P) = c(P) - \frac{c(P) \cdot u}{|u|^2} u \quad \text{for all } P \in K',$$

$$s'(P) = s(P) \quad \text{for all } P \in K'.$$

In the assumption that (2) holds, the conversion to an MH problem is a bit more complicated. In the corresponding reduced MH problem $[d', D'; V'; K', c', s']$ we set

$$d' = d, \qquad D' = D - 1, \qquad V' = V \cup \{u\}, \qquad K' = K,$$

$$c'(P) = c(P) - \frac{c(P) \cdot u}{|u|^2} u \quad \text{for all } P \in K'.$$

The construction of the function $s'(P)$, for all $P \in K$, is somewhat more delicate. Suppose that in the original MH problem, P is such that $c(P) \cdot u < 0$. If $s(P) = 0$, then P contributes a unit to $A(x)$ if and only if $0 \leqslant c(P) \cdot x = c(P) \cdot (y + \alpha u) = c'(P) \cdot y + \alpha c(P) \cdot u$. As long as $c'(P) \cdot y > 0$ there will exist an $\alpha > 0$ such that this inequality holds. However, if $c'(P) \cdot y \leqslant 0$ we will have $c(P) \cdot x < 0$ and the inequality will fail. Thus we can only let P contribute a unit to $A'(x)$, the maximum in the reduced problem, if $c'(P) \cdot y > 0$, and so we must set $s'(P) = 1$. A similar analysis for the other cases leads to the following set of rules for determining the function s':

$$s'(P) = \begin{cases} 0 & \text{if } s(P) = 0 \text{ and } c(P) \cdot u \geqslant 0, \\ & \text{or if } s(P) = 1 \text{ and } c(P) \cdot u > 0, \\ 1 & \text{if } s(P) = 0 \text{ and } c(P) \cdot u < 0, \\ & \text{or if } s(P) = 1 \text{ and } c(P) \cdot u \leqslant 0. \end{cases}$$

Let $y_1(u)$ and $y_2(u)$ be the solutions to the MH problems corresponding to $u \in U$ for case (1) and for case (2), respectively. Then the candidates for $x \in T$ such that $A(x) = A^*$ corresponding to $y_1(u)$ and $y_2(u)$ are given as follows.

The case (1) candidate is simply $x_1(u) = y_1(u)$. In case (2) the situation is somewhat more complicated. Here the candidate will be of the form $y_2(u) + \alpha u$, and we must choose α carefully, so as to ensure that $y_2(u) \cdot c(P) > 0$ implies $(y_2(u) + \alpha u) \cdot c(P) > 0$ for all $P \in K$. But this is fairly straightforward. Let $V =$ $\{P \in K : y_2(u) \cdot c(P) > 0 \text{ and } u \cdot c(P) < 0\}$. The desired implication will hold if

$y_2(u) \cdot c(P) > -\alpha u \cdot c(P)$ for all $P \in V$, so it will suffice to choose $\alpha = \varepsilon/\delta$, where $\varepsilon = \min\{y_2(u) \cdot c(P): P \in V\}$ and $\delta = 1 + \max\{|u \cdot c(P)|: P \in V\}$.

A solution to the original MH problem can thus, by Lemma 3.1, be found among the set $\{x_1(u): u \in U\} \cup \{x_2(u): u \in U$ and $s(P) = 1$ for some $P \in K\}$, and hence involves solving at most $2n$ MH problems of one less dimension. We thus obtain a straightforward recursive procedure, whose running time can easily be determined to be at most

$$T(n, d) = O(2^{d-1} d n^d)$$

where $T(n, d)$ is the time required to solve an MH problem with $|K| = n$ and of dimension $D = d$. For fixed $d > 1$, this is simply $O(n^d)$.

We might point out that there is a wide range of possibilities for improvements by constant factors. In particular, there is much duplication of subproblems as it stands now, since all permutations of a set of d elements of K will yield distinct subproblems even though many of these subproblems are identical. Furthermore, one could save some effort by combining two points of K when their projections coincide or lie on the same ray from the origin of $\boldsymbol{R}^d$. We leave the details of this fine tuning to those interested in actually implementing the algorithm.

We content ourselves with the presentation of a major improvement, which reduces the time to $O(n^{d-1} \log n)$, as explained in the next section.

4. An improved densest hemisphere algorithm for two dimensions

In the preceding algorithms, we have for simplicity assumed that the deepest possible level of recursion occurs for dimension $D = 1$. This also establishes the base of induction $O(nd)$ for the estimate of the running time. We now describe an $O(n \log n) + O(dn)$ algorithm for the MIXED HEMISPHERE problem with $D = 2$ which could be used at the deepest level of recursion, thereby speeding up the general algorithm for arbitrary dimension by a factor of at least $n/\log n$. A similar improvement for the CLOSED HEMISPHERE problem can be obtained in much the same way.

Let $[2, d; V; K, c, s]$ be an MH problem. Then T is a plane and $\{c(P): P \in K\}$ is a set of points in this plane: with a total work $O(nd)$ we can express these points in terms of two coordinates in T. The solution to our problem is either $\mathbf{0}$ or a point $y \in T$ with $|y| > 0$. As before, set $U = \{c(P): P \in K$ and $c(P) \neq \mathbf{0}\}$, and for each $u \in U$ let $H(u) = \{y \in T: y \cdot u = 0\}$. We observe that in this case each $H(u)$ is a straight line through the origin in the plane T. Let us think of each of these lines as two directed rays leaving the origin in opposite directions. Pick an orientation for the plane T, and label the two rays making up $H(u)$ as $R^-(u)$ and $R^+(u)$, where the three rays $R^-(u)$, u, and $R^+(u)$ will be encountered in just that order if we start at $R^-(u)$ and proceed in a counterclockwise direction (see Fig. 1). The rays in

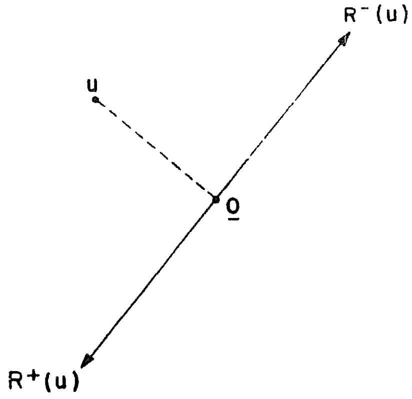

Fig. 1. The orientation of $R^-(u)$ and $R^+(u)$ in T.

$H = \{R^-(u), R^+(u)\colon u \in U\}$ divide the plane T into wedge shaped regions. To find the boundaries of these regions, we need only sort and relabel the elements of H as $r_0, r_1, \ldots, r_{2s-1}$, where $s = |U|$, so that if we start at r_0 and proceed in a counter-clockwise direction, we would meet each r_i in turn until we get back to r_0 (see Fig. 2). One way to accomplish this sorting would be to compute polar angles $\theta(r)$, $0 \leq \theta(r) < 2\pi$, for each $r \in H$ relative to some chosen r_0 with $\theta(r_0)$ set to 0, and then sort the values of $\theta(r)$. This has the apparent drawback that some of the $\theta(r)$'s may be irrational numbers. Fortunately, it is possible to determine if $\theta(r) < \theta(r')$ in constant time, without actually computing the values of θ. Let $r_0 = R^+(u_0)$ for some $u_0 \in U$, and suppose that r, r' are distinct elements of H. Then the relationship between $\theta(r)$ and $\theta(r')$ is specified as follows.

Choose u, $u' \in U$ such that $r \in \{R^-(u), R^+(u)\}$ and $r' \in \{R^-(u'), R^+(u')\}$, and let the coordinates of u and u' in T be (a, b) and (a', b'), respectively. Select points p and p' in r and r', respectively, as follows:

$$p = \begin{cases} (b, -a) & \text{if } r = R^-(u), \\ (-b, a) & \text{if } r = R^+(u), \end{cases}$$

$$p' = \begin{cases} (b', -a') & \text{if } r = R^-(u), \\ (-b', a') & \text{if } r = R^+(u'). \end{cases}$$

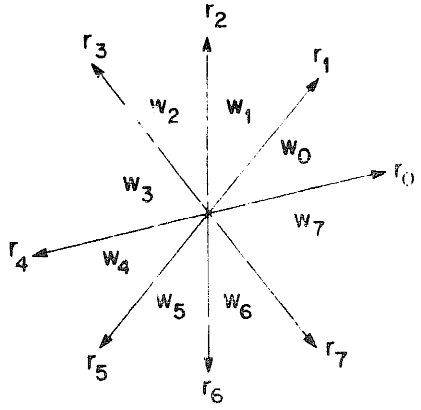

Fig. 2. The regions of T (for $|U| = 4$).

Since r and r' are distinct, we must have either $\theta(r) > \theta(r')$ or $\theta(r) < \theta(r')$.

If $p \cdot u_0 \geq 0$ and $p' \cdot u_0 < 0$, then $\theta(r) < \theta(r')$.

If $p \cdot u_0 < 0$ and $p' \cdot u_0 \geq 0$, then $\theta(r) > \theta(r')$.

If $p \cdot u_0 = p' \cdot u_0 = 0$ then, if $r = r_0$ we have $\theta(r) < \theta(r')$, otherwise $\theta(r) > \theta(r')$.

If none of the above hold, then $\theta(r) > \theta(r')$ unless

(i) $r = R^-(u)$ and $p' \cdot u > 0$, or

(ii) $r = R^+(u)$ and $p' \cdot u < 0$.

Thus, using standard sorting algorithms we can determine our desired ordering $r_0, r_1, \ldots, r_{2s-1}$ of H in time $O(n \log n)$. Let W_i be the region bounded by r_i and $r_{i+1(\text{mod } 2s)}$. As before, we note that $A(x)$ will be constant on each of the convex regions W_i, with possible discontinuities on the boundaries. There are thus essentially $4s + 1$ different candidates for an x which maximizes $A(x)$, one for each ray r_i, one for each region W_i, and one for $\mathbf{0}$. To be specific, choose a non-zero point p_i in each ray r_i, $0 \leq i \leq 2s-1$. Then $q_i = p_i + p_{i+1(\text{mod } 2s)}$ will be a point in W_i, $0 \leq i \leq 2s-1$. The value of $A(x)$ must be maximized by some point in $\{p_i, q_i : 0 \leq i \leq 2s-1\} \cup \{\mathbf{0}\} = C$.

We can evaluate $A(\mathbf{0})$ and $A(p_0)$ in time $O(nd)$. The remainder of the values can be computed in time $O(nd)$ *overall*, as follows. Suppose $A(p_i)$ has been computed for some i, $0 \leq i \leq 2s-1$. Then

$$\begin{aligned} A(q_i) = A(p_i) &+ |\{P \in K : c(P) \neq \mathbf{0}, s(P) = 1, \text{ and } R^-(c(P)) = r_i\}|, \\ &- |\{P \in K : c(P) \neq \mathbf{0}, s(P) = 0, \text{ and } R^+(c(P)) = r_i\}|. \end{aligned}$$

If $A(q_i)$ has been computed for some i, $0 \leq i \leq 2s-1$, then

$$\begin{aligned} A(p_{i+1}) = A(q_i) &+ |\{P \in K : c(P) \neq \mathbf{0}, s(P) = 0, \text{ and } R^-(c(P)) = r_{i+1}\}|, \\ &- |\{P \in K : c(P) \neq \mathbf{0}, s(P) = 1, \text{ and } R^+(c(P)) = r_{i+1}\}|. \end{aligned}$$

Since each $P \in K$ is encountered at most twice in this procedure, the overall time is $O(nd)$. Finding that $x \in C$ with maximum $A(x)$ now requires only $O(n)$ time. The total time needed to solve the MH problem with $D = 2$ is thus dominated by the time for sorting H, and is $O(n \log n) + O(nd)$ as claimed.

Using this procedure as the final step in the recursion of Section 3 thus gives an algorithm for the OPEN HEMISPHERE problem on n points and d dimensions with running time at most $O(d2^{d-2}n^{d-1} \log n)$. The analogous algorithm for the CLOSED HEMISPHERE problem has running time bounded by $O(dn^{d-1} \log n)$.

5. Closing remarks

In this paper we have shown that both the closed and open HEMISPHERE problems are NP-complete when the number of dimensions d is not fixed in advance. However, for fixed $d \geq 2$, we have described algorithms for determining a densest hemisphere which require a number of operations at most $O(n^{d-1} \log n)$.

It is worth pointing out that the described techniques are directly applicable to an interesting generalization of the problem, in which each $P \in K$ is weighted through a function $w : k \to \mathbf{Q}$. For instance, in the WEIGHTED MIXED HEMISPHERE problem, we must seek an $x \in T$ which maximizes

$$A(x) = \sum_{P \in W(x)} w(P)$$

where

$$W(x) = \{P \in K : s(P) = 0 \text{ and } x \cdot c(P) \geq 0\}$$
$$\cup \{P \in K : s(P) = 1 \text{ and } x \cdot c(P) > 0\}.$$

It is easily recognized that the algorithms described in Sections 3 and 4 can be modified to solve this problem, since here again the set $U = \{c(P): P \in K\} - \{\mathbf{0}\}$ induces a partition of T into plane-bounded convex regions, in each of which the function $A(x)$ assumes a constant value.

We raise an open question whether our techniques can be modified to solve the problem of finding a P^* whose induced hemispheres partition the set K most equally.

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